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The Energy Level Spacing for Two Harmonic Oscillators with Golden Mean Ratio of Frequencies

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Berry and Tabor discussed, among other things, a beautiful problem about the energy level spacing distribution for a system of two harmonic oscillators. They gave some interesting theoretical arguments which show that there is no level clustering for generic harmonic oscillators, and various numerical experiments were exposed and discussed. But the main question they posed about the existence of the limit distribution of the level spacing remained open. The present paper discusses this question in the case when the ratio of the frequencies is the golden mean $\sigma = (\sqrt{5}-1)/2$. The approach enables one to study the generic case of the frequency ratio as well, which is done elsewhere.

KEY WORDS: Quantum linear oscillators; distribution of energy level spacing.

We consider the distribution of the energy level spacing for a system of two harmonic quantum oscillators. The Hamiltonian of the model is

$$H = \frac{p_1^2 + \omega_1^2 q_1^2}{2} + \frac{p_2^2 + \omega_2^2 q_2^2}{2}$$

The energy levels are

$$E_{mn} = m\omega_1 + n\omega_2 + E_{00}, \qquad m, n \geqslant 0$$

We are interested in the distribution of the distances between neighbor levels E_{mn}, $E_{m'n'}$ for large E_{mn}. Since

$$E_{mn} = \omega_1(m + \alpha n) + E_{00}$$

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where $\alpha = \omega_2/\omega_1$, the problem is reduced to the similar one for the system

$$\{\lambda_{mn} = m + \alpha n,\ m, n \geqslant 0\}$$

We shall assume that α is irrational, $0 < \alpha < 1$.

Expand α into the continued fraction,

$$\alpha = [a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

and put

$$\frac{p_j}{q_j} = [a_1, a_2, \ldots, a_j]$$

In that follows we shall consider three ensembles of $\{\lambda_{mn}\}$:

(I) Microensemble μ_k. Let $k \in \mathbf{N}$ and M_k be the set $\{\lambda_{mn} \mid k-1 \leqslant \lambda_{mn} \leqslant k\}$. Then μ_k is the uniform measure on M_k.

(II) Small ensemble $\mu^{(j)}$ is the uniform measure on $M^{(j)} = \{\lambda_{mn} \mid p_{j-1} \leqslant \lambda_{mn} \leqslant p_j\}$.

(III) Grand ensemble $\mu(\Lambda)$ is the uniform measure on $M(\Lambda) = (\lambda_{mn} \mid 0 \leqslant \lambda_{mn} \leqslant \Lambda\}$.

We start with the microensemble. All our subsequent considerations are based on the following simple proposition. Let $\{x\} = x - \{x\}$ denote the fractional part of x, $\{x\} \in S^1 = \mathbf{R}/\mathbf{Z}$.

Proposition 1. The set of distances between neighbor levels $\lambda_{mn} = m + \alpha n$ in the interval $k - 1 \leqslant m + \alpha n \leqslant k$ coincides with the set of distances between neighbors in the set $\{\{n\alpha\}, 0 \leqslant n < k/\alpha\}$.

Remark. The set $\{\{n\alpha\}, 0 \leqslant n < k/\alpha\}$ is considered on the circle $S^1 = \mathbf{R}/\mathbf{Z}$.

Proof. Since $m + \alpha n \leqslant k$, m, $n \geqslant 0$, then $n < k/\alpha$. Besides, if $k - 1 \leqslant m + \alpha n < k$ and $k - 1 \leqslant m' + \alpha n' < k$, then

$$m + \alpha n - (m' + \alpha n') = \{m + \alpha n\} - \{m' + \alpha n'\} = \{n\alpha\} - \{n'\alpha\}$$

and

$$k + \alpha 0 - (m + \alpha n) = 1 - \{n\alpha\}$$

which proves the proposition.

Now we shall study the distribution of the neighbor distances in the set $\{\{n\alpha\},\ 0 \leqslant n < l\}$, $l = [k/\alpha]$. We shall consider the case when α is the "golden mean" $(\sqrt{5}-1)/2$, but the construction can be extended to other cases as well.

Denote

$$\sigma = \frac{\sqrt{5}-1}{2}, \qquad \tau = \frac{\sqrt{5}+1}{2}$$

so that $q_{1,2} = -\sigma, \tau$ are the roots of the equation $q^2 - q - 1 = 0$. Let $f_1, f_2, f_3, \dots$ be the sequence of Fibonacci numbers 1, 1, 2, 3, 5,... . Then

$$\sigma = [1, 1, \dots], \qquad \frac{f_{j-1}}{f_j} = [1, 1, \dots, 1] \quad (j \text{ 1's})$$

Put

$$\varepsilon_j = |f_j \sigma - f_{j-1}|$$

As well known, for $j \to \infty$,

$$\frac{f_j}{f_{j-1}} = \tau + O(\sigma^j)$$

and

$$\frac{\varepsilon_{j-1}}{\varepsilon_j} = \tau$$

Besides, $\varepsilon_{j-1} = \varepsilon_j + \varepsilon_{j+1}$.

Proposition 2. If $f_j < l \leqslant f_{j+1}$, then the distances ε between neighbor elements in the set $\{\{n\sigma\},\ 0 \leqslant n < l\}$ can be only ε_{j-2}, ε_{j-1}, or ε_j. The numbers l_i of the neighbors with the distance ε_i, $i = j-2, j-1, j$, are equal to

$$l_{j-2} = f_{j+1} - l$$
$$l_{j-1} = l - f_{j-1}$$
$$l_j = l - f_j$$

The proof of this well-known proposition is rather simple and we omit it. Remark only that when one adds the point $\{l\sigma\}$ to the set $\{\{n\sigma\},\ 0 \leqslant n < l\}$ it splits some segment of the length ε_{j-2} on the circle into two segments of length ε_{j-1} and ε_j. This process is continued until $l = f_{j+1}$, when all the segments of the length ε_{j-2} are exhausted. Next the process begins of splitting the segments of length ε_{j-1} into segments of the length ε_j and ε_{j+1}, and so on.

The neighbor distances ε_{j-2}, ε_{j-1}, ε_j go to 0 when $j \to \infty$, so to study their limit distribution, we should normalize them. It is convenient to normalize them by the least distance ε_j. To this end, introduce normalized distances $s = \varepsilon/\varepsilon_j$,

$$s_{j-2} = \frac{\varepsilon_{j-2}}{\varepsilon_j} = \tau^2$$

$$s_{j-1} = \frac{\varepsilon_{j-1}}{\varepsilon_j} = \tau$$

$$s_j = 1$$

The limit distribution of the normalized distances in the microensemble μ_k as $k \to \infty$ is described by the following statement.

Theorem 3. For $k = f_{j-1} + x f_{j-2}$, $0 < x \leqslant 1$, the limit distribution of normalized neighbor distances in the microensemble μ_k as $j \to \infty$ is located at the points $\{1, \tau, \tau^2\}$ and the weight of these points are $x/(\tau + x)$, $(\tau - 1 + x)/(\tau + x)$, and $(1 - x)/(\tau + x)$, respectively.

Proof. Denote by ε the distance from a given point in M_k to the right neighbor. We have

$$f_j < l = \left[\frac{k}{\sigma}\right] < f_{j+1}$$

so by Proposition 2.1, ε takes values ε_{j-2}, ε_{j-1}, ε_j and

$$\Pr\{s=1\} = \Pr\{\varepsilon = \varepsilon_j\} = \frac{l - f_j}{l} = \frac{k - f_{j-1}}{k} + O(\sigma^j)$$

$$= \frac{x f_{j-2}}{f_{j-1} + x f_{j-2}} + O(\sigma^j) = \frac{x}{\tau + x} + O(\sigma^j)$$

so

$$\lim_{j \to \infty} \Pr\{s = 1\} = \frac{x}{\tau + x}$$

which was stated. Similarly, one has

$$\Pr\{s=\tau\} = \lim_{j\to\infty} \Pr\{\varepsilon=\varepsilon_{j-1}\} = \lim_{j\to\infty} \frac{l-f_{j-1}}{l} = \lim_{j\to\infty} \frac{k-f_{j-2}}{k}$$

$$= \lim_{j\to\infty} \frac{f_{j-1}-(1-x)\,f_{j-2}}{f_{j-1}+xf_{j-2}} = \frac{\tau-1+x}{\tau+x}$$

$$\Pr\{s=\tau^2\} = \lim_{j\to\infty} \Pr\{\varepsilon=\varepsilon_{j-2}\} = \lim_{j\to\infty} \frac{f_{j+1}-l}{l} = \lim_{j\to\infty} \frac{f_j-k}{k}$$

$$= \lim_{j\to\infty} \frac{f_{j-2}-xf_{j-2}}{f_{j-1}+xf_{j-2}} = \frac{1-x}{\tau+x}$$

Theorem 3 is proved.

Theorem 3 shows that the distribution of normalized neighbor distances in the ensemble μ_k is asymptotically, when $k\to\infty$, a periodic function of $\log k$ with the period $\log \tau$.

The limit distribution in the small ensemble is described in the next theorem.

Theorem 4. The limit distribution of normalized neighbor distances in the small ensemble $\mu^{(j)}$ as $j\to\infty$ is located at the points $\{1, \tau, \tau^2\}$ and the weights of these points are $2\sigma-1$, $3-4\sigma$, and $2\sigma-1$, respectively.

Proof. We have

$$\lim_{j\to\infty} \Pr\{\varepsilon=\varepsilon_j\} = \lim_{j\to\infty} \frac{\sum_{f_j<l<f_{j+1}} (l-f_j)}{\sum_{f_j<l<f_{j+1}} l} = \lim_{j\to\infty} \frac{f_{j+1}-f_j}{f_{j+1}+f_j}$$

$$= \frac{\tau-1}{\tau+1} = 2\sigma-1$$

$$\lim_{j\to\infty} \Pr\{\varepsilon=\varepsilon_{j-1}\} = \lim_{j\to\infty} \frac{\sum_{f_j<l<f_{j+1}} (l-f_{j-1})}{\sum_{f_j<l<f_{j+1}} l} = \lim_{j\to\infty} \frac{f_{j-2}+f_j}{f_{j+1}+f_j}$$

$$= \frac{1+\tau^2}{\tau^2+\tau^3} = 3-4\sigma$$

$$\lim_{j\to\infty} \Pr\{\varepsilon=\varepsilon_{j-2}\} = \lim_{j\to\infty} \frac{\sum_{f_j<l<f_{j+1}} (f_{j+1}-l)}{\sum_{f_j<l<f_{j+1}} l} = \lim_{j\to\infty} \frac{f_{j-1}}{f_{j+1}+f_j}$$

$$= \frac{\tau-1}{\tau+1} = 2\sigma-1$$

which proves the theorem.

Theorem 5. The limit distribution of normalized neighbor distances in the grand ensemble $\mu(f_{j-1}+xf_{j-2})$, where $0\leqslant x\leqslant 1$, as $j\to\infty$, is located at the points $\{1, \tau, \tau^2, \tau^3,...\}$ and the weights of these points are: $\Pr\{1\}=x^2/(\tau+x)^2$, $\Pr\{\tau\}=(\tau-1+x)^2/(\tau+x)^2$, $\Pr\{\tau^2\}=(1+2x-x^2)/(\tau+x)^2$, and $\Pr\{\tau^s\}=2\tau^{-2(s-2)}/(\tau+x)^2$ for $s\geqslant 3$.

Proof. We have

$$\lim_{j\to\infty}\Pr\{\varepsilon=\varepsilon_j\}$$

$$=\lim_{j\to\infty}\frac{\sum_{f_j<l<f_j+xf_{j-1}}(j-f_j)}{\sum_{0<l<f_j+xf_{j-1}}l}=\lim_{j\to\infty}\frac{(xf_{j-1})^2}{(f_j+xf_{j-1})^2}$$

$$=\left(\frac{x}{\tau+x}\right)^2$$

$$\lim_{j\to\infty}\Pr\{\varepsilon=\varepsilon_{j-1}\}$$

$$=\lim_{j\to\infty}\frac{\sum_{f_{j-1}<l<f_j+xf_{j-1}}(l-f_{j-1})}{\sum_{0<l<f_j+xf_{j-1}}l}$$

$$=\lim_{j\to\infty}\frac{[f_j-(1-x)f_{j-1}]^2}{(f_j+xf_{j-1})^2}=\left(\frac{\tau-1+x}{\tau+x}\right)^2$$

$$\lim_{j\to\infty}\Pr\{\varepsilon=\varepsilon_{j-2}\}$$

$$=\lim_{j\to\infty}\frac{\sum_{f_{j-2}<l<f_j}(l-f_{j-2})+\sum_{f_j<l<f_j+xf_{j-1}}(f_{j+1}-l)}{\sum_{0<l<f_j+xf_{j-1}}l}$$

$$=\lim_{j\to\infty}\frac{f_{j-1}^2+f_{j-1}^2-f_{j-1}^2(1-x)^2}{(f_j+xf_{j-1})^2}=\frac{1+2x-x^2}{(\tau+x)^2}$$

$$\lim_{j\to\infty}\Pr\{\varepsilon=\varepsilon_{j-s}\}$$

$$=\lim_{j\to\infty}\frac{\sum_{f_{j-s}<l<f_{j-s+2}}(l-f_{j-s})+\sum_{f_{j-s+2}<l<f_{j-s+3}}(f_{j-s+3}-l)}{\sum_{0<l<f_j+xf_{j-1}}l}$$

$$=\lim_{j\to\infty}\frac{2f_{j-s+1}^2}{(f_j+xf_{j-1})^2}=\frac{2(\tau^{2-s})^2}{(\tau+x)^2}$$

Theorem 5 is proved.

Again the distribution of normalized neighbor distances in the ensemble $\mu(\Lambda)$ is an asymptotically periodic function in $\log\Lambda$ with the period $\log\tau$. In the particular case $x=1$ we have the sequence of weights

$$\Pr\{1\}=\tau^{-4};\qquad \Pr\{\tau\}=\tau^{-2};\qquad \Pr\{\tau^s\}=2\tau^{-2s},\qquad s\geqslant 2 \qquad (*)$$

The first moment of the limit distribution is

$$m(x) \equiv \sum_{s \geqslant 0} \tau^s \Pr\{\tau^s\} = \frac{x(2\tau + 4) + 6\tau + 2}{(\tau + x)^2}$$

In particular, $m(1) = 6 - 2\tau$.

CONCLUSION

As was stressed in ref. 1, the distribution of energy level spacing in the system of two linear oscillators has some peculiar properties different from the general nonlinear case. Our calculations enable us to characterize more specifically these properties. They are: (i) rigidity of spectrum spacing; (ii) periodic behavior of the spacing distribution; (iii) nonuniversality of the limit spacing distribution; and (iv) long tail of the limit spacing distribution.

The manifestation of the rigidity of the spectrum spacing is its *discreteness*, which is connected with the *high multiplicity* of each admitted spacing: The multiplicity is proportional to the whole number of levels! The probability of each level spacing, which is the fraction of this spacing among all the possible ones, has no limit as $\Lambda \to \infty$, but it *fluctuates periodically* in $\log \Lambda$, where Λ is the length of the spectral interval under consideration. The limit, however, does exist when one takes it by the sequence of the Fibonacci numbers $\Lambda = f_j$, $j \to \infty$.

The limit *is not universal*, in the sense that it depends on α. To see this, one can repeat our calculations for $\alpha = \sqrt{2} - 1 = [2, 2, 2,...]$. Let $p_n/q_n = [2, 2,..., 2]$ (n 2's). Then for $\Lambda = p_n \to \infty$ the limit spacing distribution exists and it is given by the same formula ($*$) but with $\tau = \sqrt{2} + 1$. The nonuniversality actually has much stronger character and for any *generic* α one has no limit at all of the spacing distribution even for the subsequence $\Lambda = p_n \to \infty$. What one can hope to prove is the existence of this limit *in distribution* when $0 \leqslant \alpha \leqslant 1$ is random and it obeys an absolutely continuous distribution on [0, 1]. We are going to consider this interesting question elsewhere.

Remark, finally, that the limit spacing distribution ($*$) has a long tail, so that the second moment of the distribution diverges. Some considerations show that this property is typical and for any generic α the spacing distribution has a long tail.

NOTE ADDED IN PROOF

Just before receiving the proofs of this article the author received a letter from Jean Bellissard, in which his attention was called to the paper [2]. In that important paper it was shown that locally for any irrational α the distance between neighbor energy levels can take only three values and the local spacing distribution is fluctuating so that no limit distribution exists. (Compare with Theorem 3 of this article.)

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